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Some Coincidence Theorems in Wedges, Cones, and Convex Sets

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1. INTRODUCTION

In this paper we study existence of solutions to nonlinear equations of the form $Lx = Nx$ where L is not necessarily invertible. We use coincidence degree results to obtain solutions in a prescribed wedge, cone, or convex set. Some applications will be discussed in Section 4.

In Section 2 we state a theorem of Gaines and Santanilla [3], and obtain as a corollary a generalization of the Schauder's fixed point theorem when Ker L is not necessarily trivial.

In Section 3 we establish coincidence degree results of compression type generalizing some theorems of Leggett and Williams [8] and Krasnosel'skii [7].

Section 4 is devoted to illustrating some of our results. We shall obtain solutions to the following problems:

$$\dot{x}(t) = f(t, x(t))$$
$$x(0) = x(1)$$
$$x(t) \geqslant 0, \qquad x \not\equiv 0$$

and

$$-\ddot{x}(t) = f(t, x(t))$$
$$x(0) = x(1) = 0$$
$$x(t) \geqslant 0.$$

2. A COINCIDENCE THEOREM OF SCHAUDER'S TYPE

We start by introducing some basic notation relative to coincidence degree (see [2, 9]). Let X and Z be real Banach spaces. We shall consider a linear

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mapping $L\colon \operatorname{dom} L \subset X \to Z$ and a not necessarily linear mapping $N\colon X \to Z$ with the following properties:

(i) L is a Fredholm operator of index zero, i.e., $\dim(\operatorname{Ker} L) = \operatorname{codim}(\operatorname{Im} L) < \infty$ and $\operatorname{Im} L = L(\operatorname{dom} L)$ is closed. It follows now from basic results of linear functional analysis that there exist continuous projectors $P\colon X \to X$ and $Q\colon Z \to Z$ such that $\operatorname{Im} P = \operatorname{Ker} L$ and $\operatorname{Ker} Q = \operatorname{Im} L$. Since $\dim(\operatorname{Im} Q) = \operatorname{codim}(\operatorname{Im} L)$ there is an isomorphism $J\colon \operatorname{Im} Q \to \operatorname{Ker} L$. Further, the restriction L_p of L to $\operatorname{dom} L \cap \operatorname{Ker} P$ is one-to-one and onto $\operatorname{Im} L$, so that its (algebraic) inverse $K_p\colon \operatorname{Im} L \to \operatorname{dom} L \cap \operatorname{Ker} P$ is defined. It is well known that $Lx = \lambda Nx$ is equivalent to

$$x = Px + JQNx + \lambda K_p(I - Q)\,Nx$$

for all $\lambda \in (0, 1]$.

(ii) N is L-completely continuous, i.e., the mappings QN and $K_p(I - Q)\,N\colon X \to X$ are completely continuous on every bounded subset of X.

Let C be a nonempty closed convex subset of X. $\gamma\colon X \to C$ will denote a continuous retraction, i.e., $\gamma|_C = I$, and Ω will denote a nonempty open bounded subset of X. We assume that γ maps subsets of $\bar{\Omega}$ into bounded subsets. Finally, we define $M \equiv P + JQN + K_p(I - Q)\,N$ and $\tilde{M} \equiv M \circ \gamma$.

The following result gives conditions under which the equation $Lx = Nx$ has a solution in the convex set C.

THEOREM 2.1 [3]. *Let the following conditions be satisfied*:

(A) $(P + JQN)\,\gamma(\partial\Omega) \subset C$ *and* $\tilde{M}(\bar{\Omega}) \subset C$;

(B) $Lx \neq \lambda Nx$ *for every* $x \in C \cap \partial\Omega \cap \operatorname{dom} L$ *and* $\lambda \in (0, 1)$;

(C) $d_B[[I - (P + JQN)\gamma]|_{\operatorname{Ker} L}, \operatorname{Ker} L \cap \Omega, 0] \neq 0$ (*Brouwer degree*).

Then $Lx = Nx$ *has a solution* $x \in C \cap \bar{\Omega}$.

COROLLARY 2.2. *Suppose the following*:

(i) $N\gamma(\bar{\Omega}) \in L(C)$;

(ii) $Lx \neq \lambda Nx$ for every $x \in C \cap \partial\Omega \cap \operatorname{dom} L$ and $\lambda \in (0, 1)$;

(iii) $0 \in \Omega \cap C$;

(iv) $\operatorname{Ker} L = \{0\}$.

Then $Lx = Nx$ *has a solution in* $C \cap \bar{\Omega}$.

Proof. From assumption (iv) it follows that $P \equiv 0$ and $Q \equiv 0$. Since $0 \in C$, condition (A) is satisfied. Further, we note that

$$d_B[I - (P + JQN)\,\gamma|_{\operatorname{Ker} L}, \operatorname{Ker} L \cap \Omega, 0] = d_B[I, \{0\}, 0] = 1 \neq 0.$$

Observation. If Ω is convex, by taking $C = \bar{\Omega}$ in Corollary 2.2 we have condition (ii) automatically and we obtain a theorem of Schauder's type when the $\operatorname{Ker} L$ is trivial. In particular, when $X = Z$ and $L = I$, this result reduces to the Schauder's fixed point theorem for a convex closed set with nonempty interior. The theorem of Schauder's type when the $\operatorname{Ker} L$ is trivial also follows from Corollary IV.11 in [9]. For $\operatorname{Ker} L$ not necessarily trivial we have the following

COROLLARY 2.3. *Let Ω be convex. Assume that the following conditions are satisfied*:

(i) $(P + JQN)(\partial\Omega) \subset \Omega$ *and* $M(\bar{\Omega}) \subset \bar{\Omega}$;

(ii) $d_B[JQN|_{\operatorname{Ker} L}, \operatorname{Ker} L \cap \Omega, 0] \neq 0$.

Then $Lx = Nx$ has a solution in $\bar{\Omega}$.

Proof. Let us take $C = \bar{\Omega}$ in Theorem 2.1. Suppose $Lx = \lambda Nx$ *for some* $\lambda \in (0, 1)$ and $x \in \partial\Omega \cap \operatorname{dom} L$. We have

$$x = (P + JQN)\, x + \lambda K_p(I - Q)\, Nx$$
$$= (1 - \lambda)(P + JQN)\, x + \lambda Mx,$$

hence $x \in \Omega$. This contradiction implies that hypothesis (B) in Theorem 2.1 is satisfied, and the result follows.

Remark 1. If $X = Z$, $L = I$ and $0 \in \Omega$, Corollary 2.3 is the Schauder's fixed point theorem. A different generalization of this theorem when $\operatorname{Ker} L \neq \{0\}$ may be obtained from Corollary 3.2 in [10].

Remark 2. The original proof of Theorem 2.1 was given with $(P + JQN)\gamma\ (\bar{\Omega}) \subset C$ in hypothesis (A) instead of $(P + JQN)\gamma\ (\partial\Omega) \subset C$. The modification is trivial.

3. Coincidence Theorems of Compression Type

In many mathematical models, such as population and epidemic equations, it is important to have results assuring nonnegative solutions. This can be accomplished by reducing the model to an operator equation having "nonnegative" solutions, i.e., solutions in some wedge or cone. In this section we shall establish coincidence results of compression type for solutions in a wedge or cone. Expansion theorems are obtained in a similar fashion.

A map $N: K \to K$ of the cone K of an ordered Banach space is said to be a *compression of* K if $N(0) = 0$ and if there exist numbers $R > r > 0$ such that

$$x - Nx \notin K \qquad \text{if} \quad x \in K,\ \|x\| \leqslant r, \text{and } x \neq 0 \tag{3.1}$$

and, for all $\varepsilon > 0$,

$$Nx - (1+\varepsilon)x \notin K \qquad \text{if} \quad x \in K \text{ and } \|x\| \geqslant R. \tag{3.2}$$

Krasnosel'skii [7, p. 137] has shown that if N is a compression of the cone K and is completely continuous on K, then N has at least one fixed point x in K with $r \leqslant \|x\| \leqslant R$. Further, it can be shown (see [12, pp. 16–17]) that Krasnosel'skii's compression theorem remains valid if (3.1) and (3.2) are replaced by the weaker conditions

$$x - Nx \notin K \qquad \text{if} \quad x \in K \text{ and } \|x\| = r \tag{3.1$'$}$$

and

$$x \in K, \|x\| = R, x = \lambda Nx \Rightarrow \lambda \geqslant 1. \tag{3.2$'$}$$

In a recent paper, Leggett and Williams [8] have improved the compression theorem by replacing (3.1′) with the less restrictive condition

$$x - Nx \notin K \qquad \text{if } x \in K(u) \text{ and } \|x\| = r, \tag{3.3$'$}$$

where u is fixed element of $K\backslash\{0\}$ and

$$K(u) = \{x \in K : \alpha x - u \in K \text{ for some positive number } \alpha\}.$$

To illustrate the advantage in restricting attention to a set of the form $K(u)$, Leggett and Williams applied their theorem to the nonlinear equation

$$x(t) = \int_{t-\tau}^{t} f(s, x(s))\, ds,$$

improving results in [1, 13, 11, 14].

The following theorem, for solutions in a wedge $W \subset X$, generalizes the result of Leggett and Williams [8, Theorem 2] and Theorem 4.6 in [12]. Using the notation of the previous section with γ being a retraction of W, we have

THEOREM 3.1. *Let Ω_1 be a nonempty open bounded subset of X and assume that the following conditions hold*:

(i) $Lx \neq \lambda Nx$ *for every* $(x, \lambda) \in (W \cap \partial\Omega_1 \cap \operatorname{dom} L) \times (0, 1]$;

(ii) $d_B[[I - (P + JQN)\gamma]|_{\operatorname{Ker} L}, \operatorname{Ker} L \cap \Omega_1, 0] \neq 0$;

(iii) *there exists a nonempty open set* Ω_2

with $\bar{\Omega}_2 \subset \Omega_1 \subset X$, *and* $u \in W\backslash\{0\}$ *such that* $x - Mx \notin W$ *for all* $x \in W(u) \cap \partial\Omega_2$;

(iv) $(P + JQN)\,\gamma(\partial\Omega_1) \subset W$ *and* $\tilde{M}(\bar{\Omega}_1 \backslash \Omega_2) \subset W$.

Then $Lx = Nx$ *has a solution in* $W \cap (\Omega_1 \backslash \Omega_2)$.

Proof. This is based on the property of invariance under homotopy of the Leray–Schauder degree. Let

$$\tilde{M}(x, \lambda) = (P + JQN)\,\gamma x + \lambda K_p(I - Q)\,N\gamma x$$

for $(x, \lambda) \in \Omega_1 \times [0, 1]$. We first show that $x \neq \tilde{M}(x, \lambda)$ for $(x, \lambda) \in \partial\Omega_1 \times (0, 1]$. If this is not the case, $x = (1 - \lambda)(P + JQN)\,\gamma x + \lambda \tilde{M} x \in W$. Thus $x = Px + JQNx + \lambda K_p(I - Q)\,Nx$. Since this equation is equivalent to $Lx = \lambda Nx$ we reach a contradiction with hypothesis (i). From the implicit assumption that in (ii) the Brouwer degree is well defined, it follows that $x \neq \tilde{M}(x, 0)$ for every $x \in \partial\Omega_1$. Hence, by the property of invariance under homotopy,

$$\begin{aligned} d[I - \tilde{M}(\cdot, 1), \Omega_1, 0] &= d[I - \tilde{M}(\cdot, 0), \Omega_1, 0] \\ &= d[I - (P + JQN)\,\gamma, \Omega_1, 0] \\ &= d_B[[I - (P + JQN)\,\gamma]|_{\mathrm{Ker}\,L}, \mathrm{Ker}\,L \cap \Omega_1, 0] \neq 0. \end{aligned}$$

Next we assert that $d[I - \tilde{M}(\cdot, 1), \Omega_2, 0] = 0$. Let β be chosen such that $\beta\,\|u\| > \sup_{x \in \bar{\Omega}_2} \|x - \tilde{M}(x, 1)\|$ and consider the family of mappings $I - \tilde{M}(\cdot, 1) - \lambda\beta u$ for $\lambda \in [0, 1]$. We may assume that $\tilde{M}(\cdot, 1)$ has no fixed points in $\partial\Omega_2$ (otherwise the proof of the theorem is complete). This in conjunction with assumption (iii) implies that $x \neq \tilde{M}(x, 1) + \lambda\beta u$ for every $(x, \lambda) \in \partial\Omega_2 \times [0, 1]$. Hence, $d[I - \tilde{M}(\cdot, 1), \Omega_2, 0] = d[I - \tilde{M}(\cdot, 1) - \beta u, \Omega_2, 0] = 0$.

Thus,

$$\begin{aligned} & d[I - \tilde{M}(\cdot, 1), \Omega_1 \backslash \bar{\Omega}_2, 0] \\ &\quad = d[I - \tilde{M}(\cdot, 1), \Omega_1, 0] - d[I - \tilde{M}(\cdot, 1), \Omega_2, 0] \neq 0. \end{aligned}$$

Finally, from the second part of hypothesis (iv) we conclude that M has a fixed point in $W \cap (\Omega_1 \backslash \bar{\Omega}_2)$ and the proof is now complete.

Remark 3.2. When $u \in \mathrm{Ker}\,L$ in Theorem 3.1, we obtain a similar result. Indeed, the conclusion of this theorem still holds if (iii) is replaced by

(iii′) There exists $u \in (W\backslash\{0\}) \cap \mathrm{Ker}\,L$ such that

$$Lx + QNx \neq Nx \qquad \text{for all } x \in W(u) \cap \partial\Omega_2 \cap \mathrm{dom}\,L.$$

The proof of this statement is very similar to that of Theorem 3.1. It is obvious that condition (iii′) is not satisfied if $\operatorname{Ker} L = \{0\}$. To allow a trivial kernel and obtain another generalization to the result of Leggett and Williams quoted above, we require $u \in \operatorname{Ker} P$. In fact, the conclusion of Thereom 3.1 holds if (iii) is replaced by

(iii″) There exists $u \in (W \backslash \{0\}) \cap \operatorname{Ker} P$ such that

$$Lx - Nx \notin L(W \cap \operatorname{Ker} P) \qquad \text{for all } x \in W(u) \cap \partial\Omega_2 \cap \operatorname{dom} L.$$

The use of hypothesis (iii′) will be illustrated in the next section.

In proving our next theorem we shall need the following:

LEMMA 3.3. *Let H be a compact subset of a cone $K \subset X$ with $0 \notin H$. Then 0 does not belong to the closed convex hull of H.*

Proof. See [12, p. 9].

THEOREM 3.4. *Let Ω_1, Ω_2 be nonempty open bounded subsets of X with $\bar{\Omega}_2 \subset \Omega_1$, and let K be a cone in X. Let us assume that conditions* (i), (ii) *and* (iv) *of Theorem* 3.1 *hold with W replaced by K. Further suppose that*

(a) *$\lambda x \neq Mx$ for every $(x, \lambda) \in (\partial\Omega_2 \cap K) \times (0, 1)$;*

(b) $\inf_{x \in \partial\Omega_2} \|\tilde{M}x\| > 0$.

Then $Lx = Nx$ has a solution in $K \cap (\Omega_1 \backslash \Omega_2)$.

Proof. As in the proof of Theorem 3.1, conditions (i), (ii) and (iv) imply that $d[I - \tilde{M}(\cdot, 1), \Omega_1, 0] \neq 0$. We will use conditions (a) and (b) to show that $d[I - \tilde{M}(\cdot, 1), \Omega_2, 0] = 0$.

Let

$$M_2 : X \to \operatorname{Conv} \tilde{M}(\partial\Omega_2)$$

be a completely continuous operator satisfying $M_2 x = \tilde{M}x$ on $\partial\Omega_2$. By assumption (b), $0 \notin H = \tilde{M}(\partial\Omega_2)$, which together with Lemma 3.3 implies that 0 is not in the closed convex hull of H. Hence $\inf_{x \in X} \|M_2 x\| = \alpha > 0$. Now let us take $k > \max\{1, r/\alpha\}$ where $r > 0$ is such that $\|x\| \leqslant r$ for every $x \in \bar{\Omega}_2$, and consider the family of mappings

$$(1 - \lambda)\tilde{M}x + \lambda k M_2 x$$

for $(x, \lambda) \in \bar{\Omega}_2 \times [0, 1]$. Assuming that $\tilde{M}$ has no fixed points in $\partial\Omega_2$ and using condition (a) we conclude that

$$x \neq (1 - \lambda)\tilde{M}x + \lambda k M_2 x$$

for every $(x, \lambda) \in (\partial\Omega_2) \times [0, 1]$. Therefore

$$d[I - \tilde{M}(\cdot, 1), \Omega_2, 0] = d[I - kM_2, \Omega_2, 0] = 0,$$

and the proof is complete.

Let $\Omega_1 = \{x \in X: \|x\| < R\}$, $\Omega_2 = \{x \in X: \|x\| < r\}$ and $D = K \cap (\bar{\Omega}_1 \backslash \Omega_2)$ where r and R are real numbers with $0 < r < R$. Let $N: D \to K$, $L = I$ and $X = Z$, then Theorem 3.4 reduces to Theorem 1.2 in [4] attributed to Krasnosel'skii by [5]. Smith [13] has recently shown that several of the known compression theorems are consequences of Theorem 1.2 in [4] where some related results are established.

Degree theory has been used in [12] to show compression and expansion fixed point theorems. The arguments we employed in proving that $d[I - \tilde{M}(\cdot, 1), \Omega_2, 0] = 0$ are in the spirit of [12].

4. Applications

We first consider the problem

$$\dot{x}(t) = f(t, x(t)) \tag{4.1}$$

$$x(0) = x(1) \tag{4.2}$$

where $f: [0, 1] \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous and $f(0, \cdot) = f(1, \cdot)$. We seek nonzero solutions satisfying $x(t) \geqslant 0$ on $[0, 1]$.

In order to apply Theorem 3.1 we define appropriate operators associated with (4.1)–(4.2). Let

$$X = \{x: [0, 1] \to \mathbb{R}^n; x \text{ is continuous and } x(0) = x(1)\}$$

$$Z = X \text{ with } \|x\| = \sup_{t \in [0,1]} \|x(t)\|$$

$$L: \operatorname{dom} L \to Z, \qquad x \mapsto \dot{x}, \qquad \text{where } \operatorname{dom} L = \{x \in X: \dot{x} \text{ is continuous}\}$$

$$N: X \to Z, \qquad x \mapsto f(\cdot, x(\cdot)).$$

We note that

$$\operatorname{Ker} L = \{x \in \operatorname{dom} L: x(t) = c \in \mathbb{R}^n \text{ for all } t \in [0, 1]\}$$

$$\operatorname{Im} L = \left\{z \in Z: \int_0^1 z(s)\, ds = 0\right\} \quad (\operatorname{Im} L \text{ is closed})$$

and

$$\dim \operatorname{Ker} L = \operatorname{codim} \operatorname{Im} L.$$

Thus L is linear and Fredholm of index zero. We define

$$P: X \to \operatorname{Ker} L, \qquad x \longmapsto \int_0^1 x(s)\, ds$$

and

$$Q: Z \to Z, \qquad z \longmapsto \int_0^1 z(s)\, ds.$$

It is easily seen that P and Q induce the structure described in the beginning of Section 2. In particular, for $z \in \operatorname{Im} L$,

$$(K_p z)(t) = \int_0^1 G(s, t)\, z(s)\, ds$$

where

$$G(s, t) = \begin{cases} s + 1, & 0 \leqslant s < t \\ s, & t \leqslant s \leqslant 1. \end{cases}$$

We make the following assumptions concerning f:

(H_1) There exist $R > 0$ and α with $0 \leqslant \alpha \leqslant \frac{2}{3}$ such that $f(t, x) \geqslant -\alpha x$ for $x \geqslant 0$ and $\|x\| \leqslant R$.

(H_2) $x \cdot f(t, x) < 0$ for all $x \geqslant 0$ with $\|x\| = R$.

(H_3) There exists a nonnegative Lebesgue integrable function $a(t)$ such that for each $k \in (0, 2)$ there exists $r > 0$ sufficiently small satisfying

$$f(t, x) \geqslant k a(t)\, x$$

for all $t \in [0, 1]$ and all $x \geqslant 0$ with $\|x\| \leqslant r$.

We are now prepared to establish the existence of nonnegative solutions to problem (4.1)–(4.2).

THEOREM 4.1. *Let* (H_1)–(H_3) *be satisfied and suppose that there exists a finite collection* $\{I_1, I_2, ..., I_N\}$ *of intervals of* $[0, 1]$ *such that*

$$\prod_{j=1}^{N} \int_{I_j} a(s)\, ds > 1, \tag{4.3}$$

then (4.1)–(4.2) *has a nonzero solution satisfying* $x \geqslant 0$ *and* $\|x\| < R$.

Proof. To apply Theorem 3.1 we first show that conditions (i), (ii) and (iv) are satisfied. We refer the reader to the proof of Theorem 3.1 in [3] for details.

Let $\Omega_1 = \{x \in X: \|x\| < R\}$. We define

$$W = \{x \in X: x(t) \geqslant 0 \text{ on } [0, 1]\}$$

and $\gamma: X \to W$ by

$$(x_1(t), x_2(t),..., x_n(t)) \mapsto (|x_1(t)|, |x_2(t)|,..., |x_n(t)|).$$

For convenience we will use the notation $(\gamma x)(t) = \gamma_x(t)$.

If J is the identity mapping, we have

$$(\tilde{M}x)(t) = \int_0^1 \gamma_x(s)\, ds + \int_0^1 f(s, \gamma_x(s))\, ds$$
$$+ \int_0^1 G(s, t)\left[f(s, \gamma_x(s)) - \int_0^1 f(\tau, \gamma_x(\tau))\, d\tau\right] ds$$

and hence

$$(\tilde{M}x)(t) = \int_0^1 \gamma_x(s)\, ds + \int_0^1 H(s, t) f(s, \gamma_x(s))\, ds$$

where

$$H(s, t) = \begin{cases} \frac{3}{2} - (t - s), & 0 \leqslant s < t \\ \frac{1}{2} + (s - t), & t \leqslant s \leqslant 1. \end{cases}$$

This, together with (H_1), implies that condition (iv) is satisfied.

Let $\phi: \mathbb{R}^n \cap \Omega_1 \to \mathbb{R}^n$ be defined by

$$\phi(c) = c - \gamma(c) - \int_0^1 f(s, \gamma(c))\, ds.$$

It follows from (H_2) that $d_B[\phi, \mathbb{R}^n \cap \Omega_1, 0] = 1$ and hence condition (ii) is satisfied. Condition (i) also follows from (H_2).

To establish (iii) we proceed as in the proof of Theorem 3 in [8]. Let $u(t) \equiv (1, 1,..., 1)$, and choose $k \in (0, 2)$ such that

$$\left(\frac{k}{2}\right)^N \prod_{j=1}^{N} \int_{I_j} a(s)\, ds > 1.$$

Choose $r > 0$ sufficiently small such that $R > r$ and

$$f(t, x) \geqslant k a(t)\, x,$$

for $t \in [0, 1]$ and all $x \geqslant 0$ with $\|x\| \leqslant r$. Suppose that, for some $x \in W(u)$ with $\|x\| = r$, $x - Mx \in W$ where

$$(Mx)(t) = \int_0^1 x(s)\,ds + \int_0^1 f(s, x(s))\,ds + \int_0^1 G(s,t)\left[f(s, x(s)) - \int_0^1 f(\tau, x(\tau))\,d\tau\right] ds.$$

Then

$$\begin{aligned}\int_{I_{j+1}} a(t)\,x(t)\,dt &\geqslant \int_{I_{j+1}} a(t)(Mx)(t)\,dt \\ &\geqslant \int_{I_{j+1}} a(t)\left[\int_0^1 H(s,t) f(s, x(s))\,ds\right] dt \\ &\geqslant \frac{k}{2}\left(\int_{I_{j+1}} a(t)\,dt\right)\left(\int_{I_j} a(s)\,x(s)\,ds\right).\end{aligned}$$

Hence

$$\int_{I_N} a(t)\,x(t)\,dt \geqslant \left(\frac{k}{2}\right)^N \left(\prod_{j=1}^{N} \int_{I_j} a(s)\,ds\right)\left(\int_{I_N} a(t)\,x(t)\,dt\right).$$

Since $(k/2)^N \left(\prod_{j=1}^{N} \int_{I_j} a(s)\,ds\right) > 1$ and $\int_{I_N} a(t)\,x(t)\,dt > 0$, we have reached a contradiction. Thus all the assumptions of Theorem 3.1 are satisfied and the proof is complete.

We note that Theorem 4.1 allows $f(t, x)$ to be zero for all $x \geqslant 0$ with $\|x\| = r < R$ and t in some subintervals of $[0, 1]$. This is not the case in a similar result contained in the remark following Theorem 3.2 in [3]. In fact, in that result it is assumed that $x \cdot f(t, x) > 0$ for $t \in [0, 1]$ and all $x \geqslant 0$ with $\|x\| = r$.

To end our discussion on problem (4.1)–(4.2), we state a version of Theorem 3.2 in [3].

THEOREM 4.2. *Let* $0 < r < R$ *and assume that*

(i) $f(t, x) \cdot x < 0$ *for all* $x \geqslant 0$ *with* $\|x\| = R$;

(ii) *there exists* $u_0 \in \mathbb{R}^n \setminus \{0\}$ *with* $u_0 \geqslant 0$ *such that if* $\|x\| = r$ *and* $\delta x \geqslant u_0$ *for some* $\delta > 0$, $x \cdot f(t, x) < 0$;

(iii) $f(t, x) \geqslant g(t)$ *for* $x \geqslant 0$ *and* $r \leqslant \|x\| \leqslant R$ *where* $g\colon [0, 1] \to \mathbb{R}^n$ *is a Lebesgue integrable function and*

$$(2 - t)\int_0^t g(s)\,ds + (1 - t)\int_t^1 g(s)\,ds \geqslant 0$$

for $t \in [0, 1]$. *Then* (4.1)–(4.2) *has a solution* x *satisfying* $r \leqslant \|x\| < R$ *and* $x \geqslant 0$.

Proof. This is based on Remark 3.2. Conditions (i), (ii) and (iv) of Theorem 3.1 are satisfied as in the proof of Theorem 3.2 in [3]. Let W, γ and Ω_1 as in Theorem 4.1. Let Ω_2 be the open ball in X with center at the origin and radius r. Suppose $Lx + QNx = Nx$ for some $x \in W(u_0)$ with $\|x\| = r$ and $x \in \operatorname{dom} L$. We would have

$$\dot{x}(t) = f(t, x(t)) - \int_0^1 f(s, x(s))\, ds.$$

Choose $\bar{t} \in [0, 1]$ such that $\max_{t \in [0,1]} \|x(t)\|^2 = \|x(\bar{t})\|^2 = r^2$. Then

$$\begin{aligned} 0 = \left.\frac{d\,\|x(t)\|^2}{dt}\right|_{t=\bar{t}} &= 2x(\bar{t}) \cdot (\bar{t}) \\ &= 2x(\bar{t}) \cdot \left[f(\bar{t}, x(\bar{t})) - \int_0^1 f(s, x(s))\, ds \right] \\ &\leqslant 2x(\bar{t}) \cdot \left[f(\bar{t}, x(\bar{t})) - \int_0^1 g(s)\, ds \right] \\ &\leqslant 2x(\bar{t}) \cdot f(\bar{t}, x(\bar{t})) \\ &< 0. \end{aligned}$$

This contradiction implies that (iii′) in Remark 3.2 is satisfied.

Observation. Suppose that conditions (i) and (iii) of Theorem 4.2 are satisfied, and (ii) is replaced by

(ii′) $x \cdot f(t, x) > 0$ for all $x \geqslant 0$ with $\|x\| = r$.

Then, from the remark following Therem 3.2 in [3], it is known that (4.1)–(4.2) has a solution x satisfying $r < \|x\| < R$ and $x \geqslant 0$.

It is easy to see that in Theorems 4.1 and 4.2, a condition can be omitted if we only seek nonnegative solutions (possibly zero). For example, if we omit condition (ii) in Theorem 4.2, we obtain a solution x satisfying $\|x\| < R$ and $x \geqslant 0$. For a different approach to existence of nonnegative solutions to problem (4.1)–(4.2), see Section 7.3 in [7].

Finally, we shall discuss existence of nonnegative solutions to the problem

$$-\ddot{x}(t) = f(t, x(t)) \tag{4.4}$$

$$x(0) = x(1) = 0 \tag{4.5}$$

where $f: [0, 1] \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous. If we define $X = \{x: [0, 1] \to \mathbb{R}^n; x$ is continuous and $x(0) = x(1) = 0\}$ with the usual norm.

$$Z = \{x: [0, 1] \to \mathbb{R}^n; x \text{ is continuous}\}$$

$$\text{dom } L = \{x \in X: \ddot{x} \text{ is continuous}\}$$

$$L: \text{dom } L \to Z, x \longrightarrow -\ddot{x}$$

and

$$N: X \to Z, \qquad x \mapsto f(\cdot, x(\cdot))$$

then the Picard problem may be written as $Lx = Nx$.

In establishing existence of nonnegative solutions to problem (4.4)–(4.5), we shall use the next three propositions.

PROPOSITION 4.3. *Let $x: [0, 1] \to \mathbb{R}^n$ be continuously differentiable with $x(0) = x(1) = 0$. Then $\|x\|_2 \leqslant \|\dot{x}\|_2$.*

PROPOSITION 4.4. *Let $x \in \text{dom } L$. Then*

$$|\dot{x}(t)|^2 = \frac{d}{dt}[\dot{x}(t) \cdot x(t)] - \ddot{x}(t) \cdot x(t).$$

PROPOSITION 4.5. *Let x as in Proposition 4.3. Then $\|x\| \leqslant \|\dot{x}\|_2$.*

Let $g: [0, 1] \to \mathbb{R}_+$ be a Lebesgue integrable function and $\alpha \in (0, 1)$. Let $\rho = (1 - \alpha)^{-1} \|g\|_1$ and

$$R > \rho.$$

We have

THEOREM 4.6. *Assume that the following conditions hold:*

(i) *there exist $\alpha \in (0, 1)$, and $g: [0, 1] \to \mathbb{R}_+$ a Lebesgue integrable function such that $x \cdot f(t, x) \leqslant \alpha \|x\|^2 + g(t)\|x\|$ for all $x \geqslant 0$;*

(ii) *$f(t, x) \geqslant -ax$ for all $x \geqslant 0$ with $\|x\| \leqslant R$.*

Then (4.4)–(4.5) has a nonnegative solution.

Proof. Consider the family of problems

$$-\ddot{x}(t) = -(1 - \lambda)\, ax(t) + \lambda f(t, x(t)) \tag{4.6}$$

$$x(0) = x(1) = 0 \tag{4.7}$$

for $\lambda \in (0, 1)$.

Equations (4.6)–(4.7) are equivalent to $\hat{L}x = \lambda \hat{N}x$ where $\hat{L} = L - A$, $\hat{N} = N - A$ and $Ax = -ax$. We shall apply Corollary 2.2 to the abstract equation $\hat{L}x = \hat{N}x$. We first show that any possible solution x of (4.6)–(4.7) with $\lambda \in (0, 1)$ and $x(t) \geqslant 0$ is a priori bounded. We have

$$\begin{aligned} -x(t) \cdot \ddot{x}(t) &= -(1-\lambda)\,\alpha\,\|x(t)\|^2 + \lambda x(t) \cdot f(t, x(t)) \\ &\leqslant x(t) \cdot f(t, x(t)) \\ &\leqslant \alpha\,\|x(t)\|^2 + g(t)\|x(t)\|. \end{aligned}$$

Integrating over $[0, 1]$, using Proposition 4.4 and the boundary conditions,

$$\|\dot{x}\|_2^2 \leqslant \alpha\,\|x\|_2^2 + \|g\|_1\,\|x\|.$$

From Propositions 4.3 and 4.5 it follows that

$$\|x\| \leqslant (1-\alpha)^{-1}\,\|g\|_1.$$

(In addition, we note that

$$\begin{aligned} \|\ddot{x}(t)\| &\leqslant \lambda\,\|f(t, x(t))\| + (1-\lambda)\,\alpha\,\|x(t)\| \\ &\leqslant \sup_{\substack{t \in [0,1] \\ \|x\| \leqslant \rho}} \|f(t, x)\| + \alpha\rho, \end{aligned}$$

and hence using Rolle's theorem we obtain

$$\|\dot{x}\| \leqslant \sup_{\substack{t \in [0,1] \\ \|x\| \leqslant \rho}} \|f(t, x)\| + \alpha\rho).$$

Let

$$\Omega = \{x \in X : \|x\| < R\}$$

and

$$C = \{x \in X : x(t) \geqslant 0\}.$$

It follows that $\hat{L}x \neq \lambda \hat{N}x$ for all $x \in C \cap \partial\Omega \cap \operatorname{dom} \hat{L}$ and $\lambda \in (0, 1)$.

Now we show that condition (i) of Corollary 2.2 is satisfied. Since $\operatorname{Ker} \hat{L} = \{0\}$ and $\hat{L}$ is onto,

$$\tilde{M}x = K_0(N - A)\,\gamma x$$

where γ is defined as before. Solving the problem $-\ddot{x}(t) + \alpha x(t) = z(t)$, $x(0) = x(1) = 0$ with $z \in Z$, we have

$$x(t) = \int_0^1 G(s, t)\, z(s)\, ds$$

where

$$G(s,t) = \frac{1}{\sqrt{\alpha}\sinh\sqrt{\alpha}} \begin{cases} \sinh\sqrt{\alpha}(1-t)\sinh\sqrt{\alpha}\,s, & 0 \leqslant s < t \\ \sinh\sqrt{\alpha}(1-s)\sinh\sqrt{\alpha}\,t, & t \leqslant s \leqslant 1. \end{cases}$$

If $x \in \bar{\Omega}$, $\|\gamma x(s)\| \leqslant R$, and then

$$\begin{aligned}(\tilde{M}x)(t) &= \int_0^1 G(s,t)[f(s,\gamma_x(s)) + \alpha\gamma_x(s)] \\ &\geqslant 0.\end{aligned}$$

This completes the proof.

Condition (i) has been used by Mawhin in [9, Corollary v.5] to establish existence of solutions (not necessarily nonnegative) to the problem (4.4)–(4.5). Several authors have also obtained nonnegative solutions for this problem [4, 6, 7, 12]. However, they require $f(t,x) \geqslant 0$ for $x \geqslant 0$.

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